

An interior gradient estimate for the mean curvature equation of Killing graphs

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Abstract

We extend the interior gradient estimate due to Korevaar-Simon for solutions of the mean curvature equation from the case of Euclidean hypersurfaces to the general case of Killing graphs. As an application, we prove the existence and uniqueness of radial graphs in hyperbolic space with prescribed mean curvature function and asymptotic boundary data at infinity.

1 Introduction

Gradient estimates are fundamental a priori estimates for elliptic and parabolic equations and play a key role in geometry and PDE. Here we extend the interior gradient estimate for solutions of the mean curvature equation due to Korevaar-Simon [8] (see also [9]) from the case of Euclidean hypersurfaces to the general case of Killing graphs. This has been done as a tool for the result on radial graphs in hyperbolic space discussed below and should be useful for other applications. Observe that the Main Theorem in [11] for hypersurfaces in ambient spaces $N^{n+1} = M^n \times \mathbb{R}$ is a special case of our result. See also [3] and [4] for other applications of the Korevaar-Simon method.

Let N^{n+1} denote an $(n+1)$ -dimensional Riemannian manifold carrying a non-singular complete Killing vector field Y whose orthogonal distribution is integrable. Fix a integral hypersurface M^n of the orthogonal distribution. Observe that M^n is a totally geodesic submanifold of N^{n+1} . Let $\Omega \subset M^n$ be a domain such that the flow lines of the flux $\Psi: \mathbb{R} \times M^n \rightarrow N^{n+1}$ generated by Y are complete. Notice that $\gamma = 1/\langle Y, Y \rangle$ can be seen as a function in $\bar{\Omega}$ since $Y\gamma = 0$ by the Killing equation. Moreover, the solid cylinder $\Psi(\mathbb{R} \times \bar{\Omega})$ with the induced metric has a warped product Riemannian structure $\bar{\Omega} \times_\rho \mathbb{R}$ where $\rho = 1/\sqrt{\gamma}$.

Given a function u on $\bar{\Omega}$ the associated *Killing graph* is the hypersurface

$$\text{Gr}(u) = \{\Psi(u(x), x) : x \in \bar{\Omega}\}.$$

*Partially supported by CNPq and FUNCAP.

†Partially supported by CNPq.

It was shown in [2] that $\text{Gr}(u)$ has mean curvature $H(x)$ if and only if $u \in C^2(\Omega)$ satisfies

$$\mathcal{Q}[u] = \text{div}\left(\frac{\nabla u}{w}\right) - \frac{\gamma}{w}\langle \nabla u, \bar{\nabla}_Y Y \rangle = nH \quad (1)$$

where $w = \sqrt{\gamma + |\nabla u|^2}$ and H is computed for the orientation given by the Gauss map

$$N = \frac{1}{w}(\gamma Y - \Psi_* \nabla u). \quad (2)$$

Here ∇ and div denote the gradient and divergence in M^n and $\bar{\nabla}$ the Riemannian covariant derivative in N^{n+1} .

Given $o \in \Omega$ let $r > 0$ be such that $r < i(o)$ where $i(o)$ is the injectivity radius of M^n at o . We denote by $B_r(o)$ the geodesic ball contained in Ω centered at o and radius r .

Theorem 1. *Let $u \in C^3(B_r(o))$ be a negative solution of the mean curvature equation (1). Then, there exists a constant $L = L(u(o), r, \gamma, H)$ such that $|\nabla u(o)| \leq L$.*

As an application of the above result we prove the existence and uniqueness of a radial graph in hyperbolic space \mathbb{H}^{n+1} with prescribed mean curvature and boundary data at infinity. For constant mean curvature this was achieved by Bo Guan and Spruck [1] whose proof does not work in our case of variable mean curvature. Our solution to the problem is obtained as the limit of radial graphs with prescribed mean curvature on a sequence of concentric geodesic balls exhausting \mathbb{H}^n . The existence of these Killing graphs on geodesic balls follows from the general result in [2].

Let \mathbb{H}^n be a complete totally geodesic hypersurface in \mathbb{H}^{n+1} . A fixed constant speed geodesic line ℓ orthogonal to \mathbb{H}^n at a point $o \in \mathbb{H}^n$ determines uniquely a one-parameter family of translation isometries in \mathbb{H}^{n+1} that preserve ℓ and whose Killing field Y extends the velocity vector of ℓ . These isometries extend to the ideal boundary at infinity $\partial_\infty \mathbb{H}^{n+1}$ of \mathbb{H}^{n+1} as conformal transformations belonging to the conformal structure of $\partial_\infty \mathbb{H}^{n+1}$.

Denote by $\Psi: \mathbb{R} \times \mathbb{H}^n \rightarrow \mathbb{H}^{n+1}$ the flux generated by Y and set

$$\bar{\mathbb{H}}^n = \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n.$$

Given a function $u \in C^2(\mathbb{H}^n) \cap C^0(\bar{\mathbb{H}}^n)$ its *radial graph* (Killing graph) is the hypersurface

$$\text{Gr}(u) = \{\Psi(u(x), x) : x \in \mathbb{H}^n\}$$

with asymptotic boundary given by

$$\partial_\infty \text{Gr}(u) = \overline{\text{Gr}(u)} \cap \partial_\infty \mathbb{H}^{n+1}.$$

With some abuse of language, we say that $\text{Gr}(\phi) = \{\Psi(\phi(x), x) : x \in \partial_\infty \mathbb{H}^n\}$ is the radial graph of the function $\phi = u|_{\partial_\infty \mathbb{H}^n}$.

Theorem 2. *Given $H \in C^\alpha(\mathbb{H}^n)$ with $\sup_{\mathbb{H}^n} |H| < 1$ and $\phi \in C^0(\partial_\infty \mathbb{H}^n)$ there exists a unique function $u \in C^{2,\alpha}(\mathbb{H}^n) \cap C^0(\bar{\mathbb{H}}^n)$ such that $Gr(u)$ has mean curvature H and $\partial_\infty Gr(u)$ is the radial graph of ϕ .*

The above result is a natural extension to Killing graphs in the hyperbolic space of the asymptotic Dirichlet problem for the mean curvature hypersurface PDE in a complete noncompact Riemannian manifold which, in the minimal case, has been studied in [5], [6] and [10].

2 Proof of Theorem 1

For a system of coordinates $x = (x_1, \dots, x_n)$ equation (1) reads as

$$a^{ij}u_{i;j} - R\langle \nabla \gamma, \nabla u \rangle = nHw \quad \text{where} \quad a^{ij} = \sigma^{ij} - \frac{u^i u^j}{w^2}. \quad (3)$$

Here the σ^{ij} 's are the coefficients of the metric in M^n and for simplicity we denote

$$R = \frac{\gamma + w^2}{2\gamma w^2}.$$

We define a nonnegative function $\eta(x) = g(\phi(x))$ on $B_r(o)$ where

$$g(t) = e^{C_1 t} - 1$$

for some constant $C_1 > 0$ and ϕ is given by

$$\phi(x) = \left(1 - \frac{d^2(x)}{r^2} + \frac{u(x)}{2u_0} \right)^+.$$

Here $+$ means positive part, $d(x)$ is the geodesic distance to o in M^n and $C = 1/2u_o$ where $u_o = -u(o)$. Then η vanishes outside of $B_r(o)$. To be more precise, we should replace 1 by $1 - \epsilon$ in the definition of ϕ so that η is smooth with compact support and later let ϵ tend to zero, but this is omitted for simplicity.

Let $p \in B_r(o)$ be an interior point where the function $h = \eta w$ has a maximum. In the sequel computations are done at this point without further notice. From $h_i = 0$ we have

$$\eta_i w = -\eta w_i. \quad (4)$$

Moreover, the Hessian matrix of h is negative semidefinite. Taking the trace of the product of the Hessian matrix of h with the positive definite matrix a^{ij}/w yields

$$0 \geq \frac{1}{w} a^{ij} h_{i;j} = \frac{a^{ij}}{w} (w \eta_{i;j} + 2\eta_i w_j + \eta w_{i;j}).$$

We obtain using (4) that

$$a^{ij}\eta_{i;j} + \frac{\eta}{w^2}a^{ij}(ww_{i;j} - 2w_iw_j) \leq 0. \quad (5)$$

From (2) we have

$$N^k = -\frac{u^k}{w}. \quad (6)$$

Thus,

$$w_i = \frac{\gamma_i}{2w} + \frac{u^k u_{k;i}}{w} = \frac{\gamma_i}{2w} - N^k u_{k;i}. \quad (7)$$

In the sequel, we use (4), (6) and (7) several times without further reference.

We have,

$$\begin{aligned} w_{i;j} &= \frac{\gamma_{i;j}}{2w} - \frac{\gamma_i w_j}{2w^2} - N^k_{;j} u_{k;i} - N^k u_{k;ij} \\ &= \frac{\gamma_{i;j}}{2w} + \frac{\sigma^{kl}}{w} u_{l;j} u_{k;i} - \frac{w}{\eta^2} \eta_i \eta_j - N^k u_{k;ij} \\ &= \frac{\gamma_{i;j}}{2w} + \frac{1}{w} (\sigma^{kl} - N^k N^l) u_{l;j} u_{k;i} + \frac{1}{w} N^k N^l u_{l;j} u_{k;i} - \frac{1}{w} w_i w_j - N^k u_{k;ij} \\ &= \frac{\gamma_{i;j}}{2w} + \frac{a^{kl}}{w} u_{l;j} u_{k;i} + \frac{\gamma_i \gamma_j}{4w^3} - \frac{1}{2w^2} (w_i \gamma_j + w_j \gamma_i) - N^k u_{k;ij}. \end{aligned}$$

Thus,

$$a^{ij} w_{i;j} = \frac{1}{2w} a^{ij} \gamma_{i;j} + \frac{1}{w} a^{ij} a^{kl} u_{l;j} u_{k;i} + \frac{1}{4w^3} a^{ij} \gamma_i \gamma_j + \frac{1}{w\eta} a^{ij} \eta_i \gamma_j - N^k a^{ij} u_{k;ij}. \quad (8)$$

We compute the last term of (8). The Ricci identities for the Hessian of u yield

$$u_{k;ij} = u_{i;kj} = u_{i;jk} + R^l_{kji} u_l.$$

Hence,

$$N^k a^{ij} u_{k;ij} = N^k a^{ij} u_{i;jk} - \frac{1}{w} a^{ij} R^l_{kji} u^k u_l = N^k (a^{ij} u_{i;j})_{;k} - N^k a^{ij}_{;k} u_{i;j} - \frac{1}{w} a^{ij} R^l_{kji} u^k u_l.$$

It follows that

$$N^k a^{ij} u_{k;ij} = n N^k (wH)_k + N^k (R \langle \nabla \gamma, \nabla u \rangle)_k - N^k a^{ij}_{;k} u_{i;j} - \frac{1}{w} a^{ij} R^l_{kji} u^k u_l. \quad (9)$$

We compute the first three terms of (9). For the first term, we have

$$(wH)_k = \frac{w}{\eta} (\eta H_k - H \eta_k). \quad (10)$$

Since

$$\left(\frac{\gamma + w^2}{w^2}\right)_k = \frac{\gamma_k}{w^2} - \frac{\gamma}{w^4}(\gamma_k + 2u^l u_{l;k}) = \frac{1}{w^2}\left(\gamma_k + \frac{2\gamma\eta_k}{\eta}\right)$$

and

$$\left(\frac{1}{2\gamma}\langle\nabla\gamma, \nabla u\rangle\right)_k = \left(\frac{\gamma_l}{2\gamma}\right)_{;k} u^l + \frac{\gamma^l}{2\gamma} u_{l;k} = \frac{1}{2\gamma} \left[\left(\frac{\gamma_l \gamma_k}{\gamma} - \gamma_{k;l}\right) w N^l + \gamma^l u_{l;k} \right],$$

we obtain for the second term of (9) that

$$(R\langle\nabla\gamma, \nabla u\rangle)_k = R \left[\left(\frac{\gamma_l \gamma_k}{\gamma} - \gamma_{k;l}\right) w N^l + \gamma^l u_{l;k} \right] + \frac{1}{w^2} \left(\frac{\gamma_k}{2\gamma} + \frac{\eta_k}{\eta} \right) \langle\nabla\gamma, \nabla u\rangle. \quad (11)$$

We have,

$$\begin{aligned} a_{;k}^{ij} &= -\frac{1}{w^2}(u_{;k}^i u^j + u^i u_{;k}^j) + \frac{1}{w^4}(\gamma_k - 2w N^l u_{l;k}) u^i u^j \\ &= \frac{1}{w}(u_{;k}^i - N^i N^l u_{l;k}) N^j + \frac{1}{w}(u_{;k}^j - N^j N^l u_{l;k}) N^i + \frac{1}{w^2} \gamma_k N^i N^j \\ &= \frac{1}{w} a^{il} u_{l;k} N^j + \frac{1}{w} a^{jl} u_{l;k} N^i + \frac{1}{w^2} \gamma_k N^i N^j. \end{aligned}$$

It follows that the third term of (9) is given by

$$N^k a_{;k}^{ij} u_{i;j} = \frac{2}{w} a^{ij} \left(\frac{w\eta_i}{\eta} + \frac{\gamma_i}{2w} \right) \left(\frac{w\eta_j}{\eta} + \frac{\gamma_j}{2w} \right) + \frac{1}{w^2} \gamma_k N^k N^i \left(\frac{w\eta_i}{\eta} + \frac{\gamma_i}{2w} \right). \quad (12)$$

Replacing (10), (11) and (12) into (9) yields

$$\begin{aligned} N^k a^{ij} u_{k;i;j} &= n \frac{w}{\eta} N^k (\eta H_k - H \eta_k) - \frac{2}{w} a^{ij} \left(\frac{w\eta_i}{\eta} + \frac{\gamma_i}{2w} \right) \left(\frac{w\eta_j}{\eta} + \frac{\gamma_j}{2w} \right) \\ &\quad - \frac{1}{w^2} \gamma_k N^k N^i \left(\frac{w\eta_i}{\eta} + \frac{\gamma_i}{2w} \right) + \frac{1}{w^2} N^k \left(\frac{\gamma_k}{2\gamma} + \frac{\eta_k}{\eta} \right) \langle\nabla\gamma, \nabla u\rangle \\ &\quad + R \left[\left(\frac{\gamma}{2w} \sigma^{kl} + w N^k N^l \right) \frac{\gamma_k \gamma_l}{\gamma} - w N^k N^l \gamma_{k;l} + \frac{w}{\eta} \gamma_l \eta^l \right] - \frac{1}{w} a^{ij} R_{kji}^l u^k u_l. \end{aligned}$$

It now follows from (8) that

$$\begin{aligned} a^{ij} w_{i;j} - \frac{2}{w} a^{ij} w_i w_j &= \frac{3}{4w^3} a^{ij} \gamma_i \gamma_j + \frac{1}{w} a^{ij} a^{kl} u_{l;j} u_{k;i} + \frac{3}{w\eta} a^{ij} \gamma_i \eta_j + \frac{1}{2w} a^{ij} \gamma_{i;j} \\ &\quad + \frac{1}{w} a^{ij} R_{kji}^l u^k u_l - n N^k \frac{w}{\eta} (\eta H_k - H \eta_k) + \frac{1}{w^2} \gamma_k N^k N^i \left(\frac{w\eta_i}{\eta} + \frac{\gamma_i}{2w} \right) \\ &\quad - \frac{1}{w^2} N^k \left(\frac{\gamma_k}{2\gamma} + \frac{\eta_k}{\eta} \right) \langle\nabla\gamma, \nabla u\rangle - R \left[\left(\frac{\gamma}{2w} \sigma^{kl} + w N^k N^l \right) \frac{\gamma_k \gamma_l}{\gamma} - w N^k N^l \gamma_{k;l} + \frac{w}{\eta} \gamma_l \eta^l \right]. \end{aligned}$$

After multiplying both sides by η/w and discarding the non-negative terms, we obtain

$$\begin{aligned} \frac{\eta}{w} \left(a^{ij} w_{i;j} - \frac{2}{w} a^{ij} w_i w_j \right) &\geq \left[-n N^k H_k - \frac{\gamma_k}{2\gamma w^3} N^k \langle \nabla \gamma, \nabla u \rangle + \frac{1}{2w^2} a^{ij} \gamma_{i;j} \right. \\ &\quad \left. - R \left(\left(\frac{\gamma}{2w^2} \sigma^{kl} + N^k N^l \right) \frac{\gamma_k \gamma_l}{\gamma} - N^k N^l \gamma_{k;l} \right) + \frac{1}{w^2} a^{ij} R_{kji}^l u^k u_l \right] \eta \\ &\quad + \left[\left(nH + \frac{1}{w^2} N^k \gamma_k - \frac{1}{w^3} \langle \nabla \gamma, \nabla u \rangle \right) N^i + \left(\frac{3}{w^2} a^{ij} - R \sigma^{ij} \right) \gamma_j \right] \eta_i. \end{aligned}$$

It is easy to check that there is a positive constant $M = M(\gamma, H)$ such that

$$\frac{1}{w^2} \eta a^{ij} (w w_{i;j} - 2 w_i w_j) \geq -M\eta - A^i \eta_i \quad (13)$$

where $-A^i$ denotes the coefficient of η_i . It follows from (5) and (13) that

$$a^{ij} \eta_{i;j} - M\eta - A^i \eta_i \leq 0. \quad (14)$$

On the other hand,

$$\eta_i = g'(-r^{-2}(d^2)_i + C u_i)$$

and

$$\eta_{i;j} = g'(-r^{-2}(d^2)_{i;j} + C u_{i;j}) + g''(-r^{-2}(d^2)_i + C u_i)(-r^{-2}(d^2)_j + C u_j).$$

Hence,

$$\begin{aligned} a^{ij} (r^{-2}(d^2)_i - C u_i)(r^{-2}(d^2)_j - C u_j) &= \frac{C^2 \gamma}{w^2} |\nabla u|^2 - \frac{2C\gamma}{r^2 w^2} \langle \nabla u, \nabla d^2 \rangle + \frac{1}{r^4} \left(|\nabla d^2|^2 - \frac{1}{w^2} \langle \nabla u, \nabla d^2 \rangle^2 \right) \\ &\geq \frac{C^2 \gamma}{w^2} \left(|\nabla u|^2 - \frac{2}{Cr^2} \langle \nabla u, \nabla d^2 \rangle \right) \end{aligned}$$

and

$$a^{ij} (-r^{-2}(d^2)_{i;j} + C u_{i;j}) = -r^{-2} a^{ij} (d^2)_{i;j} + C(nHw + R \langle \nabla \gamma, \nabla u \rangle)$$

where

$$a^{ij} (d^2)_{i;j} = \Delta d^2 - \frac{1}{w^2} \langle \nabla_{\nabla u} \nabla d^2, \nabla u \rangle.$$

It follows from (14) that

$$\begin{aligned} \frac{C^2 \gamma}{w^2} \left[|\nabla u|^2 - \frac{2}{Cr^2} \langle \nabla u, \nabla d^2 \rangle \right] g'' + \left[\frac{1}{r^2 w^2} \langle \nabla_{\nabla u} \nabla d^2, \nabla u \rangle - \frac{\Delta d^2}{r^2} + C(nHw + R \langle \nabla \gamma, \nabla u \rangle) \right] g' \\ \leq Mg + A^i (-r^{-2}(d^2)_i + C u_i) g'. \end{aligned}$$

Since

$$-A^i (d^2)_i = \left(nH + \frac{1}{w^2} N^k \gamma_k - \frac{1}{w^3} \langle \nabla \gamma, \nabla u \rangle \right) N^i (d^2)_i + \left(\frac{3}{w^2} a^{ij} - R \sigma^{ij} \right) \gamma_j (d^2)_i$$

and

$$A^i u_i = \left(nHw + \frac{1}{w} N^k \gamma_k - \frac{1}{w^2} \langle \nabla \gamma, \nabla u \rangle \right) \frac{|\nabla u|^2}{w^2} + \frac{3}{w} a^{ij} \gamma_j N_i + R \langle \nabla \gamma, \nabla u \rangle,$$

we conclude that

$$\frac{C^2 \gamma}{\gamma + |\nabla u|^2} \left(|\nabla u|^2 - \frac{2}{Cr^2} \langle \nabla u, \nabla d^2 \rangle \right) g'' + P g' - M g \leq 0$$

where the coefficient

$$\begin{aligned} P = & \frac{n}{w} C H \gamma - \frac{1}{r^2} \left(\Delta d^2 - \frac{1}{w^2} \langle \nabla_{\nabla u} \nabla d^2, \nabla u \rangle \right) - C \left(w N^k \gamma_k - \langle \nabla \gamma, \nabla u \rangle \right) \frac{|\nabla u|^2}{w^4} \\ & - \frac{3}{w} C a^{ij} \gamma_j N_i - \frac{1}{r^2 w^3} \left(n H w^3 + w N^k \gamma_k - \langle \nabla \gamma, \nabla u \rangle \right) N^i (d^2)_i - \frac{1}{r^2} \left(\frac{3}{w} a^{ij} - R \sigma^{ij} \right) \gamma_i (d^2)_j \end{aligned}$$

of g' is bounded above by a positive constant $C_0 = C_0(u(o), r, \gamma, H)$.

Suppose that

$$|\nabla u| \geq \frac{8}{Cr} = \frac{16u_0}{r}.$$

Then, we have

$$|\nabla u| \geq \frac{4|\nabla d^2|}{Cr^2}$$

and

$$|\nabla u|^2 - \frac{2}{Cr^2} \langle \nabla u, \nabla d^2 \rangle \geq |\nabla u|^2 - \frac{2}{Cr^2} |\nabla u| |\nabla d^2| \geq \frac{1}{2} |\nabla u|^2.$$

Thus, there exists a constant $D = D(u(0), r, \gamma)$ such that

$$\frac{C^2 \gamma}{\gamma + |\nabla u|^2} \left(|\nabla u|^2 - \frac{2}{Cr^2} \langle \nabla u, \nabla d^2 \rangle \right) \geq \frac{C^2 \gamma |\nabla u|^2}{2(\gamma + |\nabla u|^2)} \geq D > 0.$$

For instance, setting $\gamma_0 = \inf_{B_r(o)} \gamma$ we may take $D = 32\gamma_0/(r^2\gamma_0 + 256u^2(0))$. Since $g(t) = e^{C_1 t} - 1$, we obtain a contradiction taking $C_1 = C_1(u(o), r, \gamma, H)$ sufficiently large, that is, such that

$$Dg''(p) + C_0 g'(p) - M g(p) > 0.$$

Hence,

$$w(p) \leq C_2 = \gamma_1 + \frac{16u_0}{r}$$

where $\gamma_1 = \sup_{B_r(o)} \gamma$. Since $h(o) \leq h(p)$, we obtain

$$\eta(o)w(o) \leq \eta(p)w(p).$$

Therefore,

$$(e^{C_1/2} - 1)w(o) \leq C_2 e^{C_1}.$$

This gives the desired estimate and concludes the proof.

Remark 3. The constant L given by Theorem 1 also depends on the geometry of M^n along $B_r(o)$ including the Ricci curvature.

3 Proof of Theorem 2

The proof of Theorem 2 will be done in two steps. First, we will show the existence of a uniform height estimate for the solutions of the Dirichlet problem on a sequence of concentric geodesic balls exhausting \mathbb{H}^n . Then, we prove that a subsequence of these solutions converges to a solution of our problem.

We represent the hyperbolic space \mathbb{H}^{n+1} of constant sectional curvature -1 as the warped product manifold

$$\mathbb{H}^{n+1} = \mathbb{H}^n \times_{\cosh \rho} \mathbb{R}$$

where ρ is the geodesic distance in \mathbb{H}^n to the point $o \in \mathbb{H}^n$ determined by the geodesic ℓ . In terms of the notation fixed earlier $Y = \partial/\partial s$ where s parametrizes the factor \mathbb{R} . In (1) we thus have

$$\gamma = 1/\cosh^2 \rho. \quad (15)$$

Let Ω_k denote the geodesic disc in \mathbb{H}^n of radius ρ_k centered at o where

$$\rho_k = \operatorname{arctanh}(1 - 1/k), \quad k = 2, 3, \dots$$

Let $F \in C^{2,\alpha}(\mathbb{H}^n) \cap C^0(\bar{\mathbb{H}}^n)$ be such that $F|_{\partial_\infty \mathbb{H}^n} = \phi$ and set $M_0 = |F|_0$. We claim that the unique solution $u_k \in C^2(\bar{\Omega}_k)$ for any $k \geq 2$ obtained in [2] of the auxiliary Dirichlet problem

$$\begin{cases} \mathcal{Q}[u_k] = nH|_{\Omega_k}, \\ u_k|_{\partial\Omega_k} = F_k = F|_{\partial\Omega_k} \end{cases}$$

has a uniform height estimate, that is, independent of k .

We recall from [2] that the Killing cylinder \mathcal{H}_k over $\partial\Omega_k$ is the hypersurface in \mathbb{H}^{n+1} ruled by the flow lines of Y through $\partial\Omega_k$, i.e.,

$$\mathcal{H}_k = \{\Psi(s, x) : s \in \mathbb{R}, x \in \partial\Omega_k\}.$$

The geodesic distance d from $\partial\Omega_k$ measured along normal geodesics pointing towards o is

$$d = \rho_k - \rho.$$

The geodesic curvature of the flow lines of Y is given by

$$\kappa = \langle \nabla d, \bar{\nabla}_{\sqrt{\gamma}Y} \sqrt{\gamma}Y \rangle = -\gamma \langle Y, \bar{\nabla}_{\nabla d} Y \rangle = \frac{\gamma'}{2\gamma} = \tanh \rho \quad (16)$$

where $'$ denotes derivative with respect to d and we used (15). Since the mean curvature of the geodesic ball $\partial\Omega_k$ is $\coth \rho_k$, then the constant mean curvature H_k of \mathcal{H}_k is

$$H_k = \frac{1}{n}((n-1)\coth \rho_k + \tanh \rho_k) > 1 \quad (17)$$

with respect to the pointing inward orientation.

Consider on Ω_k the function

$$v_k(x) = M_0 + h(d(x))$$

where h has to be chosen. Then,

$$\begin{aligned} \mathcal{Q}[v_k] &= \operatorname{div} \left(\frac{h' \nabla d}{\sqrt{\gamma + h'^2}} \right) - \frac{\gamma h'}{\sqrt{\gamma + h'^2}} \langle \nabla d, \bar{\nabla}_Y Y \rangle \\ &= \frac{h'}{\sqrt{\gamma + h'^2}} (\Delta d - \kappa) + \left(\frac{h'}{\sqrt{\gamma + h'^2}} \right)'. \end{aligned}$$

Since $-\frac{1}{n-1} \Delta d$ is the mean curvature of the geodesic sphere in \mathbb{H}^n of radius $\rho = \rho_k - d$, we have that

$$\Delta d - \kappa = -nH_k. \quad (18)$$

Thus,

$$\mathcal{Q}[v_k] = -\frac{nh'H_k}{\sqrt{\gamma + h'^2}} + \frac{\gamma(h'' - \kappa h')}{(\gamma + h'^2)^{\frac{3}{2}}}.$$

We choose

$$h(d) = C(\arcsin(\tanh \rho_k) - \arcsin(\tanh \rho))$$

where $C > 0$ is a constant to be chosen and $\arcsin : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus

$$h' = C / \cosh \rho \quad \text{and} \quad h'' = C \tanh \rho / \cosh \rho.$$

It follows from (16) that $h'' - \kappa h' = 0$ and using (15) that

$$\frac{h'}{\sqrt{\gamma + h'^2}} = \frac{C}{\sqrt{1 + C^2}}.$$

Therefore,

$$\mathcal{Q}[v_k] = -\frac{C}{\sqrt{1 + C^2}} nH_k.$$

Hence, in order to obtain that

$$Q[v_k] < Q[u_k] = nH$$

it suffices to choose C such that

$$\frac{C}{\sqrt{1 + C^2}} H_k > |H|. \quad (19)$$

By assumption $H_0 = \sup |H| < 1$. Using (17) it follows that (19) holds if we choose $C > 0$ such that

$$C > \sqrt{\frac{H_0^2}{1 - H_0^2}} \geq \sqrt{\frac{H_0^2}{H_k^2 - H_0^2}} \geq \sqrt{\frac{H^2}{H_k^2 - H^2}}.$$

We have for $x \in \partial\Omega_k$ that

$$v_k(x) \geq F_k(x).$$

We conclude that v_k is an upper barrier to u_k , that is,

$$u_k(x) \leq v_k(x), \quad x \in \Omega_k.$$

This implies that the functions u_k have a uniform height estimate since

$$v_k(x) \leq M = M_0 + C\pi,$$

and proves the claim.

It follows from the above height estimate and the gradient estimate in Theorem 1 that for given $m \in \mathbb{N}$ the sequence $\{u_k|_{\bar{\Omega}_m}\}_{k>2m}$ has equibounded C^1 norm in $\bar{\Omega}_m$. To be able to use Theorem 1 we have to translate the solutions by $-M$. By elliptic theory this sequence has equibounded $C^{2,\alpha}$ norm in Ω_m . Then, there is a subsequence $\{u_{k_j^m}\}_j$ of $\{u_k\}_{k>2m}$ converging on Ω_m in the C^2 norm to a function $v_m \in C^2(\Omega_m)$. Denoting $Q_H[u] = \mathcal{Q}[u] - nH$, we have that $Q_H[v_m] = 0$. It follows from PDE regularity (cf. [7]) that $v_m \in C^{2,\alpha}(\Omega_m)$.

Consider a subsequence $\{u_{k_j^{m+1}}\}_j$ of $\{u_{k_j^m}\}_j$ which converges in Ω_{m+1} to a function $v_{m+1} \in C^{2,\alpha}(\Omega_{m+1})$ satisfying $Q_H[v_{m+1}] = 0$. After iterating this process, we obtain a subsequence $\{u_{k_{1+i}^{m+i}}\}_i$ of $\{u_k\}_k$ that converges uniformly on compact subsets of \mathbb{H}^n in the C^2 norm to a function $u \in C^2(\mathbb{H}^n)$ satisfying $Q_H[u] = 0$. By simplicity, we also denote this subsequence by $\{u_k\}_{k \in \mathbb{N}}$. We claim that u extends continuously to $\partial_\infty \mathbb{H}^n$ and that $u|_{\partial_\infty \mathbb{H}^n} = \phi$.

We first prove that $\partial_\infty \text{Gr}(u) \subset \text{Gr}(\phi)$ by showing that if $p \in \partial_\infty \mathbb{H}^{n+1} \setminus \text{Gr}(\phi)$ then $p \notin \partial_\infty \text{Gr}(u)$. Consider $p \in \partial_\infty \mathbb{H}^{n+1} \setminus \text{Gr}(\phi)$. Since $\text{Gr}(\phi)$ is compact and $p \notin \text{Gr}(\phi)$ there exists an equidistant hypersurface E of \mathbb{H}^{n+1} such that $\partial_\infty E$ separates p from $\text{Gr}(\phi)$, that is, p and $\text{Gr}(\phi)$ are in distinct open connected components of $\partial_\infty \mathbb{H}^{n+1} \setminus \partial_\infty E$. Moreover, since $H_0 = \sup_{\mathbb{H}^n} |H| < 1$, we may assume that E has constant mean curvature H_0 with respect to the unit normal vector field pointing to the connected component U of $\mathbb{H}^{n+1} \setminus E$ whose asymptotic boundary contains $\text{Gr}(\phi)$. Set $G_k = \text{Gr}(u_k)$. By the convergence of $u_k|_{\partial\Omega_k}$ to ϕ there is k_0 such that $\partial G_k \subset U$ and then $\partial G_k \cap E = \emptyset$ for all $k \geq k_0$. By the tangency principle $G_k \cap E = \emptyset$, i.e., $G_k \subset U$ for all $k \geq k_0$. It follows that $p \notin \partial_\infty \text{Gr}(u)$.

Now consider a sequence $x_k \in \mathbb{H}^n$ converging to $x \in \partial_\infty \mathbb{H}^n$. By the compactness of $\bar{\mathbb{H}}^{n+1}$ there is a subsequence $\Psi(u(x_{k_j}), x_{k_j})$ of $\Psi(u(x_k), x_k)$ converging to $z \in \bar{\mathbb{H}}^{n+1}$. Since x_k diverges and $\Psi(u(x_{k_j}), x_{k_j}) \in \text{Gr}(u)$ it follows that $z \in \partial_\infty \text{Gr}(u)$. From what

we proved above $z \in \text{Gr}(\phi)$ and hence $z = \Psi(\phi(x_0), x_0)$ for some $x_0 \in \partial_\infty \mathbb{H}^n$. Since u is globally bounded, the sequence $\{u(x_{k_j})\}_j \subset \mathbb{R}$ is bounded and thus contains a subsequence $\{u(x_{k_{j_i}})\}_i$ converging to some $t_0 \in \mathbb{R}$. Being the extension of Ψ_{t_0} to $\bar{\mathbb{H}}^{n+1}$ continuous, we obtain that

$$z = \lim_i \Psi(u(x_{k_{j_i}}), x_{k_{j_i}}) = \Psi(t_0, x)$$

and then $\Psi(\phi(x_0), x_0) = \Psi(t_0, x)$. Since $\Psi: \mathbb{R} \times \partial_\infty \mathbb{H}^n \rightarrow \partial_\infty \mathbb{H}^{n+1}$ is injective it follows that $x = x_0$ and $t_0 = \phi(x_0)$. Because the limits are the same for any convergent subsequence considered, it follows that $u(x_k) \rightarrow \phi(x)$ as $k \rightarrow \infty$, and this proves the claim.

Finally, uniqueness follows from the maximum principle applied to the difference of two solutions, and this concludes the proof of the theorem.

References

- [1] B. Guan and J. Spruck. *Hypersurfaces of constant mean curvature in hyperbolic space with prescribed asymptotic boundary at infinity*. Amer. J. Math. **122** (2000), 1039–1060.
- [2] M. Dajczer, P. Hinojosa and J. H. de Lira. *Killing graphs with prescribed mean curvature*. Calc. Var. Partial Differential equations **33** (2008), 231–248.
- [3] K. Ecker and G. Huisken. *Interior estimates for hypersurfaces moving by mean curvature*. Invent. math. **105** (1991), 547–569.
- [4] M. Eichmair. *The Plateau problem for marginally outer trapped surfaces*. J. Diff. Geometry **83** (2009), 551–583.
- [5] N. do Espírito-Santo, S. Fornari and J. Ripoll. *The Dirichlet problem for the minimal hypersurface equation in $M \times \mathbb{R}$ with prescribed asymptotic boundary*. J. Math. Pures Appl. **93** (2010), 204–221.
- [6] J. Gálvez and H. Rosenberg. *Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces*. Amer. J. of Math. **132** (2010), 1249–1273.
- [7] D. Gilbarg and N. Trudinger. “Elliptic partial differential equations of second order.” Springer Verlag, Berlin-Heidelberg, 2001.
- [8] N. Korevaar. *An easy proof of the interior gradient bound for solutions to the prescribed mean curvature problem*. Proc. Symp. Pure Math. **45** (1986), 81–89.
- [9] N. Korevaar. *A priori interior gradient bounds for solutions to elliptic Weingarten equations*. Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), 405–421.

- [10] J. Ripoll and M. Telichevesky, *On the asymptotic Dirichlet problem for some divergence form quasi-linear elliptic PDE's*. Preprint.
- [11] J. Spruck. *Interior gradient estimates and existence theorems for constant mean curvature graphs in $M \times \mathbb{R}$* . Pure and Applied Mathematics Quarterly **3** (2007), 785–800.

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